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AN INTEGRAL EQUATION DESCRIBING CONJUGATE TRANSIENT HEAT TRANSFER IN FLUID FLOW THROUGH INSULATED PIPES

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NOMENCLATURE

a,	inside radius of pipe; half distance between parallel plates;
$A(x, t)$,	function defined by equation (37);
b,	outside radius of pipe, thickness of plate;
$B(x, \xi, t, \eta)$,	function defined by equation (38);
$C(x, t)$,	function defined by equation (39);
$D(x, t, \eta, \lambda, \beta)$,	function defined by equation (40);
$E(r, \gamma; k, j)$,	function defined by equation (16);
h,	surface heat-transfer coefficient at interface;
j,	an integer index;
$J_k(x)$,	Bessel function of first kind;
k,	thermal conductivity, also an integer index;
$K_k(x, \xi, t, \eta, n)$,	functions defined by equations (25) and (26);
$L_k(\ )$,	operators defined by equation (1);
m, n,	integer indices;
$P(n)$,	function defined by equation (33); see also equation (47);
$Q(n)$,	function defined by equation (34), see also equation (47);
r,	radial coordinate;
$R_k(r, t)$,	functions defined by equations (20) and (21); see also equations (43) and (45);
$S_k(r, t)$,	functions defined by equations (20) and (22); see also equations (44) and (46);
t,	time;
$T(r, x, t)$,	temperature in pipe wall;
$u_k(r, x, t)$,	auxiliary functions;
V,	fluid velocity;
x,	axial coordinate;
y, y_1, y_2,	transverse coordinates;
$Y_k(x)$,	Bessel function of second kind.

Greek symbols

α,	thermal (plus eddy) diffusivity;
β,	a dummy variable;
β_n,	number defined by equation (17); see also equation (48);
γ_n,	number defined by equation (19); see also equation (48);
η,	a dummy variable;
$\theta(r, x, t)$,	temperature in fluid;
λ,	a dummy variable;
ξ,	a dummy variable;
$\phi(x, t)$,	temperature of fluid at interface;
$\psi(x, t)$,	temperature of pipe wall at interface;
ω_n,	number defined by equation (18);
∇^2,	Laplace operator.

PROBLEM FORMULATION

CONSIDER a uniform semi-infinite ($x \geqq 0$) cylindrical pipe of inside radius $r = a$ and outside radius $r = b$, perfectly insulated on its outside surface and containing a homogeneous fluid which moves with constant uniform velocity V ("plug" flow) in the positive x-direction. Initially the temperature $\theta(r, x, t)$ of the fluid and the temperature $T(r, x, t)$ of the pipe are both zero. Suddenly the temperatures of the end of the pipe ($x = 0$) and of the fluid entering the pipe at $x = 0$ are increased by a unit amount. It is desired to determine θ and T for $x > 0$, $t > 0$, where t denotes time.

The analysis which follows includes the effects of thermal capacity and diffusivity in each medium. Axial symmetry is postulated. A constant uniform interface surface heat-transfer coefficient h is accounted for. However, radiative heat transfer and internal heat generation are not considered. In what follows, subscript 1 refers to fluid and subscript 2 refers to pipe. Thermal conductivity is denoted by k and diffusivity by α. Eddy diffusivity as well as molecular diffusivity is included in α_1.

We define the two operators.

$$L_1(\ ) = \alpha_1\nabla^2(\ ) - V\frac{\partial}{\partial x} - \frac{\partial}{\partial t};\ L_2(\ ) = \alpha_2\nabla^2(\ ) - \frac{\partial}{\partial t} \quad \text{(1a, b)}$$

where

$$\nabla^2(\ ) = \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \quad (2)$$

and also define the two interface temperatures

$$\phi(x,t) = \theta(a,x,t);\ \psi(x,t) = T(a,x,t). \quad \text{(3a, b)}$$

Thus we are concerned with the system

$$L_1(\theta) = 0,\ L_2(T) = 0 \text{ for } x > 0, t > 0 \quad \text{(4a, b)}$$

subject to the initial conditions

$$\theta = 0,\ T = 0 \text{ for } x \geqq 0, t = 0 \quad \text{(5a, b)}$$

and the boundary conditions

$$\left.\frac{\partial T}{\partial r}\right]_{r=b} = 0 \text{ for } x \geqq 0, t \geqq 0 \quad (6)$$

$$\theta = 1,\ T = 1 \text{ for } x = 0, t > 0 \quad \text{(7a, b)}$$

$$k_1\left.\frac{\partial\theta}{\partial r}\right]_{r=a} = k_2\left.\frac{\partial T}{\partial r}\right]_{r=a} = h(\phi - \psi) \text{ for } x \geqq 0, t \geqq 0. \quad \text{(8a, b)}$$

In all these equations where r is variable, $0 \leqq r \leqq a$ for $\theta(r,x,t)$ and $a \leqq r \leqq b$ for $T(r,x,t)$.

In what follows we formulate this problem which involves two unknown functions (θ, T) each depending on three independent variables (r,x,t) as a single integral equation (36) involving only one unknown function (ϕ) which is a function of only two independent variables (x,t).

PRELIMINARY SOLUTIONS

Space available in this note permits only a brief outline of the analysis leading to the integral equation (36). We write

$$\theta = u_1 + u_2;\ T = u_3 + u_4 \quad \text{(9a, b)}$$

where the unknown functions u_i, $(i = 1,\dots,4)$ satisfy

$$L_1(u_1) = L_1(u_2) = L_2(u_3) = L_2(u_4) = 0 \text{ for } x \geqq 0, t > 0 \quad (10)$$

$$u_1 = u_2 = u_3 = u_4 = 0 \text{ for } x \geqq 0, t = 0 \quad (11)$$

$$\left.\frac{\partial u_3}{\partial r}\right]_{r=b} = \left.\frac{\partial u_4}{\partial r}\right]_{r=b} = 0 \text{ for } x \geqq 0, t \geqq 0 \quad (12)$$

$$u_1 = u_3 = 1,\ u_2 = u_4 = 0 \text{ for } x = 0, t > 0 \quad (13)$$

$$u_2(a,x,t) = \phi(x,t),\ u_4(a,x,t) = \psi(x,t) \quad (14)$$

$$u_1(a,x,t) = u_3(a,x,t) = 0 \text{ for } x > 0, t \geqq 0. \quad (15)$$

An implicit condition is that u_1 and u_2 remain finite for $r = 0$.

The solutions for the functions u_i $(i = 1,\dots,4)$ may be represented in terms of certain useful expressions defined as follows

$$E(r,\gamma;k,j) = J_k(\gamma r)Y_j(\gamma b) - Y_k(\gamma r)J_j(\gamma b) \quad (16)$$

(where J and Y denote Bessel functions)

$$\beta_n = n\text{th positive root (in ascending order) of } J_0(\beta a) = 0 \quad (17)$$

$$\omega_n^2 = \beta_n^2 + (V/2\alpha_1)^2 \quad (18)$$

γ_n = nth positive root (in ascending order) of

$$E(a,\gamma;0,1) = 0 \quad (19)$$

$$R_1(r,n) = S_1(r,n) = (1/\beta_n a)[J_0(\beta_n r)/J_1(\beta_n a)] \quad (20)$$

$$R_2(r,n) = \frac{(a/\gamma_n)E(a,\gamma_n;1,1)E(r,\gamma_n;0,1)}{a^2E^2(a,\gamma_n;0,0) - b^2E^2(b,\gamma_n;0,1)} \quad (21)$$

$$S_2(r,n) = -\frac{E(r,\gamma_n;0,1)/\gamma_n b}{E(a,\gamma_n;0,0) + (a/b)E(a,\gamma_n;1,1)} \quad (22)$$

$$X_1(x,t,n) = e^{Vx/2\alpha_1}\left[e^{-\omega_n x}\operatorname{erfc}\left(\frac{x - 2\omega_n\alpha_1 t}{2\sqrt{(\alpha_1 t)}}\right) + e^{\omega_n x}\operatorname{erfc}\left(\frac{x + 2\omega_n\alpha_1 t}{2\sqrt{(\alpha_1 t)}}\right)\right] \quad (23)$$

$$X_2(x,t,n) = e^{-\gamma_n x}\operatorname{erfc}\left(\frac{x - 2\gamma_n\alpha_2 t}{2\sqrt{(\alpha_2 t)}}\right) + e^{\gamma_n x}\operatorname{erfc}\left(\frac{x + 2\gamma_n\alpha_2 t}{2\sqrt{(\alpha_2 t)}}\right) \quad (24)$$

$$K_1(x,\xi,t,\eta,n) = \sqrt{(\alpha_1/\pi\eta)}\,\beta_n^2\,e^{V(x-\xi)/2\alpha_1}\,e^{-\eta\alpha_1\omega_n^2} \times \{\exp[-(x-\xi)^2/4\alpha_1\eta] - \exp[-(x+\xi)^2/4\alpha_1\eta]\} \quad (25)$$

$$K_2(x,\xi,t,\eta,n) = \sqrt{(\alpha_2/\pi\eta)}\,\gamma_n^2\,e^{-\eta\alpha_2\gamma_n^2}\left\{\exp\left[\frac{-(x-\xi)^2}{4\alpha_2\eta}\right] - \exp\left[\frac{-(x+\xi)^2}{4\alpha_2\eta}\right]\right\}. \quad (26)$$

In terms of these quantities, we may write the solutions as follows:

$$u_1(r,x,t) = \sum_{n=1}^{\infty} R_1(r,n)\,X_1(x,t,n) \quad (27)$$

$$u_2(r,x,t) = \sum_{n=1}^{\infty} S_1(r,n)\int_0^{\infty}\int_0^{t}\phi(\xi,t-\eta)K_1(x,\xi,t,\eta,n)\,d\eta\,d\xi \quad (28)$$

$$u_3(r,x,t) = \sum_{n=1}^{\infty} R_2(r,n)X_2(x,t,n) \quad (29)$$

$$u_4(r,x,t) = \sum_{n=1}^{\infty} S_2(r,n) \int_0^{\infty}\int_0^{t} \psi(\xi,t-\eta)K_2(x,\xi,t,\eta,n)\,d\eta\,d\xi. \quad (30)$$

Note that conditions (15) have resulted in the appearance of the functions ϕ and ψ in the solutions for u_2 and u_4, respectively. The solutions for u_1 and u_3 are fairly trivial; those for u_2 and u_4, which appear to be quite useful in themselves, involve adapting and applying a procedure due to Lowan [1, 2]. Those interested in following the details of the derivations of the solutions u_i ($i = 1, \ldots, 4$), may refer to a thesis [4] by one of the authors, noting some changes in notation from that given above. This thesis [4] also includes several other developments relating to conjugated heat transfer in fluid flowing through insulated pipes and conduits.

DERIVATION OF THE INTEGRAL EQUATION

Equations (9) give the body temperatures θ and T in terms of the functions u_i which, in turn, are given by equations (27)–(30) in terms of the unknown surface temperatures ϕ and ψ. We also have

$$\left.\frac{\partial\theta}{\partial r}\right]_{a} = -\frac{1}{a}\sum_{n=1}^{\infty}\Big[X_1(x,t,n)$$

$$\int_0^{\infty}\int_0^{t} \phi(\xi,t-\eta)K_1(x,\xi,t,\eta,n)\,d\eta\,d\xi\Big] \quad (31)$$

$$\left.\frac{\partial T}{\partial r}\right]_{r=a} = \frac{1}{a}\sum_{n=1}^{\infty}\Big[P(n)X_2(x,t,n)$$

$$+\,Q(n)\int_0^{\infty}\int_0^{t} \psi(\xi,t-\eta)K_2(x,\xi,t,\eta,n)\,d\eta\,d\xi\Big] \quad (32)$$

where

$$P(n) = \{[bE(b,\gamma_n;0,1)/aE(a,\gamma_n;0,1)] - 1\}^{-1} \quad (33)$$

$$Q(n) = [bE(a,\gamma_n;0,0)/aE(a,\gamma_n;1,1) + 1]^{-1} \quad (34)$$

and are now in a position to obtain the integral equation. From (8a) we have

$$\psi = \phi + \frac{k_1}{h}\left.\frac{\partial\theta}{\partial r}\right]_{r=a} \quad (35)$$

and combining this with (31), we have ψ entirely in terms of ϕ. Employing this expression in (32) and also in (8b), we obtain the desired result which may be written as

$$A(x,t) + \int_0^{\infty}\int_0^{t} B(x,\xi,t,\eta)\phi(\xi,t-\eta)\,d\eta\,d\xi = (k_1k_2/ha)\Big[C(x,t)$$

$$+\int_0^{t}\int_0^{\infty}\int_0^{t-\eta} D(x,t,\eta,\lambda,\beta)\phi(\lambda,t-\eta-\beta)\,d\beta\,d\lambda\,d\eta\Big] \quad (36)$$

where

$$A(x,t) = \sum_{n=1}^{\infty}[k_1X_1(x,t,n) + k_2P(n)X_2(x,t,n)] \quad (37)$$

$$B(x,\xi,t,\eta) = \sum_{n=1}^{\infty}[k_1K_1(x,\xi,t,\eta,n) + k_2Q(n)K_2(x,\xi,t,\eta,n)] \quad (38)$$

$$C(x,t) = \int_0^{\infty}\int_0^{t}\Big[\sum_{m=1}^{\infty} X_1(\xi,t-\eta,m)\Big]$$

$$\times\Big[\sum_{n=1}^{\infty} Q(n)K_2(x,\xi,t,\eta,n)\Big]\,d\eta\,d\xi \quad (39)$$

$$D(x,t,\eta,\lambda,\beta) = \int_0^{\infty}\Big[\sum_{m=1}^{\infty} K_1(\xi,\lambda,t-\eta,\beta,m)\Big]$$

$$\times\Big[\sum_{n=1}^{\infty} Q(n)K_2(x,\xi,t,\eta,n)\Big]\,d\xi. \quad (40)$$

It is likely that some of the integrations and summations involved in definitions (37)–(40) can be carried out in finite form but we have not been successful in doing so. It is also clear that the integral equation (36), which is the goal and end result of this note, is greatly simplified in case $h \to \infty$.

THE CASE OF SLAB GEOMETRY

A similar and somewhat simpler case is that in which a uniform and homogeneous fluid is flowing with constant uniform velocity V in the positive x-direction, confined between two semi-infinite ($x \geqq 0$) plates, each parallel to the zx plane, each of thickness b, each insulated on the outside (unwetted) surface, and located at a distance $2a$ from each other measured in the y-direction. Initially all temperatures are zero. Suddenly the ends ($x = 0$) of the plates and the fluid entering between the plates at $x = 0$ experience a unit increase in temperature. We take advantage of the symmetry about the mid-fluid-plane and seek $\theta(y_1, x, t)$, the temperature of the fluid, and $T(y_2, x, t)$, the temperature of the plate, for $x \geqq 0$, $t > 0$, $0 \geqq y_1 \geqq a$, $0 \geqq y_2 \geqq b$, subject to conditions: $\theta = T = 0$ for $t = 0$; $\partial\theta/\partial y_1 = 0$ for $y_1 = a$; $\partial T/\partial y_2 = 0$ for $y_2 = b$; $\theta = T = 1$ for $x = 0$, $t > 0$; and

$$k_1\left.\frac{\partial\theta}{\partial y_1}\right]_{y_1=0} = h(\phi - \psi) = -k_2\left.\frac{\partial T}{\partial y_2}\right]_{y_2=0}. \quad (41)$$

The last condition includes the influence of a uniform constant surface heat-transfer coefficient h at the interface $y_1 = y_2 = 0$, and the negative sign results from the fact that y_1 and y_2 are taken as positive in opposite directions. Equations (1a, b) are also applicable with an obvious modification in the definition of the operator $\nabla^2()$. Defining

$$\phi(x, t) = \theta(0, x, t); \psi(x, t) = T(0, x, t) \tag{42}$$

the development is almost identically parallel to that given above for the cylindrical geometry. With the evident replacement of r by y_1 or y_2, as the case may be, equations (9), (27)–(30), (36)–(40) continue to be valid, but some modifications are called for in the definitions of various other functions. We have

$$R_1(y, n) = (1/\beta_n a) \sin(\beta_n y) \tag{43}$$

$$S_1(y, n) = (-1)^n (1/\beta_n a) \cos[\beta_n(a - y)] \tag{44}$$

$$R_2(y, n) = (1/\gamma_n b) \sin(\gamma_n y) \tag{45}$$

$$S_2(y, n) = (-1)^n (1/\gamma_n b) \cos[\gamma_n(b - y)] \tag{46}$$

$$P(n) = Q(n) = 1 \tag{47}$$

where

$$\beta_n = (n - \tfrac{1}{2})(\pi/a); \quad \gamma_n = (n - \tfrac{1}{2})(\pi/b). \tag{48}$$

Definitions (18), (23)–(26) continue to be valid.

REMARKS ABOUT EQUATION (36)

Equations (36), with the accompanying definitions of functions appearing therein, is the object of this note. The fluid interface temperature $\phi(x, t)$ is the unknown function to be determined; all other results may be obtained in a straightforward way once ϕ is known. (It is obvious that a similar formulation may be made in terms of ψ, the temperature of the pipe at the interface, rather than in terms of ϕ, and also, for the case $h \to \infty$, not only does (36) simplify greatly, the right hand side vanishing in this case, but also $\psi \equiv \phi$.) We have not yet seriously undertaken the solution of (36) but venture to make the following observations for whatever use they may be. If one considers the contours, ϕ = constant, above an xt plane, it it clear that the contour $\phi = 1$ lies along the positive t-axis while the contour $\phi = 0$ lies along the positive x-axis. In general, of course, we may expect that other contours will be curved. However, we may reasonably conjecture that the most interesting features of the solution, namely the "effective speed" with which the temperature "front" propagates (cf. Munk [3]), and the rate at which this originally abrupt front "softens" as it moves down the conduit, would not be significantly altered if the contours ϕ = constant were assumed to be straight lines passing through the origin on the xt plane. With this assumption $\phi(x, t)$ would be reduced to a function of a single variable, namely arctan (x/t), and one could seek to satisfy equation (36) by any of a number of approximate methods.

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